

ON BILATERAL WEIGHTED SHIFTS IN NONCOMMUTATIVE MULTIVARIABLE OPERATOR THEORY

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ABSTRACT. We present a generalization of bilateral weighted shift operators for the noncommutative multivariable setting. We discover a notion of periodicity for these shifts, which has an appealing diagrammatic interpretation in terms of an infinite tree structure associated with the underlying Hilbert space. These shifts arise naturally through weighted versions of certain representations of the Cuntz C^* -algebras \mathcal{O}_n . It is convenient, and equivalent, to consider the weak operator topology closed algebras generated by these operators when investigating their joint reducing subspace structure. We prove these algebras have non-trivial reducing subspaces exactly when the shifts are doubly-periodic; that is, the weights for the shift have periodic behaviour, and the corresponding representation of \mathcal{O}_n has a certain spatial periodicity. This generalizes Nikolskii's Theorem for the single variable case.

In [24] and [25], we began studying versions of unilateral weighted shift operators in noncommutative multivariable operator theory. We called them *weighted shifts on Fock space* since they act naturally on the full Fock space Hilbert space. These shifts and the algebras they generate were first studied by Arias and Popescu [3] from the perspective of *weighted Fock spaces*. The basic goals of this program are to extend results from the commutative (single variable) setting and, at the same time, expose new noncommutative phenomena. In the current paper, we continue this line of investigation by presenting versions of *bilateral* weighted shift operators for the noncommutative multivariable setting. Our analysis is chiefly spatial in nature: we examine the joint reducing subspace structure for these shifts, and consider reducibility questions for the weak operator topology closed (nonselfadjoint) algebras they generate. In particular, we give a complete characterization

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of reducibility for the algebras strictly in terms of two notions of periodicity associated with these shifts. This generalizes Nikolskii's Theorem [28] for reducing subspaces of bilateral weighted shift operators on Hilbert space.

In the first section we include a short introduction to the subject. Next we present the definition and derive some basic properties for these shifts. The second section contains the main results of the paper; most importantly, a complete description of the reducing subspace structure for the shifts and the operator algebras they generate. In the final section we include examples and state some open problems.

1. INTRODUCTION

Given a positive integer $n \geq 1$, or $n = \infty$, we may consider all possible n -tuples of operators $S = (S_1, \dots, S_n)$ which act on a common Hilbert space \mathcal{H} and satisfy the relations

$$(1) \quad S_i^* S_i = I \quad \text{for } 1 \leq i \leq n \quad \text{and} \quad \sum_{i=1}^n S_i S_i^* = I.$$

When $n = 1$, the single variable case, we are simply talking about unitary operators. Distinguished amongst the collection of unitary operators on (separable) infinite-dimensional spaces is the canonical *bilateral shift* which acts on an orthonormal basis $\{\xi_k : k \in \mathbb{Z}\}$ for \mathcal{H} by $U\xi_k = \xi_{k+1}$. Bilateral shifts play a central role in operator theory generally, and specifically within the structure theory for isometries. Thus, bilateral *weighted* shifts, operators T defined on \mathcal{H} by $T\xi_k = \lambda_{k+1}\xi_{k+1}$, $\lambda_{k+1} \in \mathbb{C}$, have also been studied for some time, often providing new classes of examples and shedding light on the theory overall (see the survey article [37] for example).

On the other hand, the set of all n -tuples $S = (S_1, \dots, S_n)$ which satisfy (1) for $n \geq 2$ is extremely vast, and there perhaps may not be an analogous structure theory here. For instance, the universal C^* -algebra generated by (1) (that is, the norm closure of the orthogonal direct sum of all possible $*$ -representations determined by n -tuples satisfying (1)), called the Cuntz algebra \mathcal{O}_n [11], has such a rich representation theory that it is not possible to classify all its representations up to unitary equivalence. (It is an NGCR algebra [19].) Thus, there has been considerable recent interest in studying various subclasses of representations of \mathcal{O}_n for a number of reasons. For examples from some of the diverse perspectives we reference, in passing, work of Ball, Bratteli, Davidson, Jorgensen, Katsoulis, Pitts, Shpigel, Vinnikov, and the author [4, 5, 6, 7, 12, 13, 14, 15, 23].

While there is not a well developed structure theory for $n \geq 2$, there are some interesting results in that direction. For instance, a dilation theorem which derives from the work of Frazho [18], Bunce [10], and Popescu [29] provides a generalization of Sz.-Nagy's classical minimal dilation of a contraction to an isometry [17]. Further, Popescu [29] gives an appropriate version of the classical Wold decomposition in this arena. In each of these cases, the role of unitary operators is played by n -tuples of isometries $S = (S_1, \dots, S_n)$ which satisfy (1). Moreover, the so-called 'left creation operators' acting on Fock space Hilbert space, which also arise in theoretical physics and free probability theory, provide the appropriate generalization of unilateral shifts for this setting. They play a role in the aforementioned dilation context, and the algebras they generate give noncommutative analogues of Toeplitz-type algebras. These algebras were discovered by Popescu [30, 32], and there is now a growing body of literature on these and other related algebras. For example, see work of Arias, Davidson, Katsoulis, Muhly, Pitts, Popescu, Power, Solel, and the author [2, 12, 13, 15, 16, 21, 22, 26, 27, 31, 33, 34, 35]. Weighted versions of these shifts have been analyzed recently as well [1, 3, 24, 25].

In this paper, we present and investigate versions $T = (T_1, \dots, T_n)$ of bilateral weighted shift operators for the noncommutative multivariable setting. Specifically, our investigation will focus on the joint reducing subspace structure of the operators $T = (T_1, \dots, T_n)$; or equivalently, reducibility of the weak operator topology closed algebras $\mathfrak{S}_{\omega, \Lambda}$ they generate. From one point of view we are considering certain "weighted free semigroup algebras", if we wish to keep in line with terminology of [12, 13, 14, 15]. We derive a complete characterization of the reducing subspaces for $\mathfrak{S}_{\omega, \Lambda}$, and this gives a generalization of Nikolskii's Theorem for the single variable case [28]. The unweighted cases for these shifts, which we denote by $S = (S_1, \dots, S_n)$, may be regarded as generalizations of the canonical bilateral shift from the single variable theory. These n -tuples determine a subclass of the atomic representations of \mathcal{O}_n considered by Davidson and Pitts [15].

2. NONCOMMUTATIVE BILATERAL WEIGHTED SHIFTS

For succinctness we drop the 'multivariable' reference when referring to these shifts. We first give a description of the possible Hilbert spaces. Throughout the rest of the paper $n \geq 2$ will be a fixed positive integer or $n = \infty$, but we will behave as though n is finite.

Definition 2.1. Let \mathbb{F}_n^+ be the unital free semigroup on n noncommuting letters written as $\{1, 2, \dots, n\}$. Let Ω_+ be the collection of infinite

words in the generators of \mathbb{F}_n^+ . Given $\omega = i_1 i_2 \cdots$ in Ω_+ , define a sequence of words in \mathbb{F}_n^+ by

$$\omega_m = \begin{cases} i_1 i_2 \cdots i_m & \text{for } m \geq 1 \\ \phi & \text{for } m = 0. \end{cases}$$

The element ϕ is the unit in \mathbb{F}_n^+ , corresponding to the empty word. Let \mathbb{F}_ω denote the set of reduced words in the free group on n generators of the form $u = v\omega_m^{-1}$ for $v \in \mathbb{F}_n^+$ and some $m \geq 0$. Let $\mathcal{K}_\omega = \ell^2(\mathbb{F}_\omega)$ be the Hilbert space with orthonormal basis $\{\xi_u : u \in \mathbb{F}_\omega\}$.

For each Hilbert space \mathcal{K}_ω , there is a class of bilateral weighted shifts which naturally act on it.

Definition 2.2. An n -tuple of operators $R = (R_1, \dots, R_n)$ acting on a Hilbert space \mathcal{H} forms a *noncommutative bilateral weighted shift* if there is an $\omega \in \Omega_+$ and a unitary $U : \mathcal{H} \rightarrow \mathcal{K}_\omega$ for which there are scalars $\Lambda = \{\lambda_u\}_{u \in \mathbb{F}_\omega}$ and operators $T = (T_1, \dots, T_n)$, given by $T_i = UR_iU^*$, such that

$$(2) \quad T_i \xi_u = \lambda_{iu} \xi_{iu} \quad \text{for } u \in \mathbb{F}_\omega \quad \text{and } 1 \leq i \leq n.$$

We also define $\mathfrak{S}_{\omega, \Lambda}$ to be the unital weak operator topology closed (nonselfadjoint) algebra generated by $\{T_1, \dots, T_n\}$,

$$\mathfrak{S}_{\omega, \Lambda} = \text{WOT-Alg}\{T_1, \dots, T_n\}.$$

Notes 2.3. (i) The shifts $T = (T_1, \dots, T_n)$ we consider will be assumed to act on a space \mathcal{K}_ω as in (2).

(ii) Given $\omega \in \Omega_+$, denote the operators for the unweighted case ($\lambda_u \equiv 1$) by $S = (S_1, \dots, S_n)$, and the associated algebra by \mathfrak{S}_ω . Observe the operators $S = (S_1, \dots, S_n)$ determine a representation of \mathcal{O}_n . In fact, they form a subclass of the *atomic* representations of \mathcal{O}_n classified by Davidson and Pitts [15] in their ongoing investigation of *free semigroup algebras*, of which \mathfrak{S}_ω provides an example.

(iii) Every infinite word ω in Ω_+ is either *aperiodic* or eventually periodic in the sense that there are $u, v_0 \in \mathbb{F}_n^+$ such that $\omega = uv_0v_0v_0 \cdots$. But for our purposes, we lose no generality in focusing on the $\omega \in \Omega_+$ which are either aperiodic, or truly periodic ($u = \phi$). The reason is that given $\omega = u\omega'$ in Ω_+ with $u \in \mathbb{F}_n^+$, $|u| = k$, and $\omega' \in \Omega_+$ periodic, one can define a unitary $U : \mathcal{K}_{\omega'} \rightarrow \mathcal{K}_\omega$ by $U\xi_{v(\omega'_m)^{-1}} = \xi_{v\omega'_{m+k}}$. This unitary intertwines the associated n -tuples $T^{(\omega)}$ and $T^{(\omega_0)}$ by $U^*T_i^{(\omega)}U = T_i^{(\omega')}$, and hence the algebras $U\mathfrak{S}_{\omega, \Lambda}U^* = \mathfrak{S}_{\omega', \Lambda}$. This type of equivalence is called a *shift-tail unitary equivalence*.

(iv) The shifts $T = (T_1, \dots, T_n)$ acting on \mathcal{K}_ω trace out an infinite tree structure for \mathcal{K}_ω , where basis vectors ξ_u and weights λ_u are identified

with vertices in the tree. This point is illustrated further in the examples of Section 4. The vertex set corresponding to words $\{\omega_m^{-1} : m \geq 0\}$ is referred to as the *main branch* of \mathbb{F}_ω .

(v) We use the following notions of length for elements of \mathbb{F}_ω and \mathbb{F}_n^+ : For $u_1, u_2 \in \mathbb{F}_\omega$, $|u_1| \leq |u_2|$ means that u_1 lies at least as close as u_2 to the closest common ancestor of u_1, u_2 on the main branch of \mathbb{F}_ω . Further, given $w \in \mathbb{F}_n^+$, $|w|$ denotes the length of the word w .

(vi) Observe that for $n = 1$, the set Ω_+ consists of the single element $\omega = 111 \dots$. Hence \mathcal{K}_ω is a single two-way infinite stalk, and in Definition 2.2 we recover standard bilateral weighted shift operators.

As we are interested in describing non-degenerative reducibility of $\mathfrak{S}_{\omega, \Lambda}$, we may clearly assume the weights $\Lambda = \{\lambda_u\}_{u \in \mathbb{F}_\omega}$ are nonzero. Furthermore, the result below shows we lose no generality in assuming weights are nonnegative. Thus the following assumption on weights will be made throughout the paper:

$$\text{Assumption: } \lambda_u > 0 \quad \text{for } u \in \mathbb{F}_\omega.$$

Proposition 2.4. *Let $T = (T_1, \dots, T_n)$ be a bilateral weighted shift on \mathcal{K}_ω with weights $\Lambda = \{\lambda_u\}_{u \in \mathbb{F}_\omega}$. Then there is a unitary U in $\mathcal{B}(\mathcal{K}_\omega)$, which is diagonal with respect to the standard basis for \mathcal{K}_ω , such that*

$$(UT_1U^*, \dots, UT_nU^*)$$

is a bilateral weighted shift on \mathcal{K}_ω with weights $\Lambda' = \{|\lambda_u|\}_{u \in \mathbb{F}_\omega}$. Thus, the algebras $\mathfrak{S}_{\omega, \Lambda}$ and $\mathfrak{S}_{\omega, \Lambda'}$ are unitarily equivalent.

Proof. We define the unitary U by $U\xi_v = \mu_v\xi_v$ upon inductively choosing scalars μ_v for $v \in \mathbb{F}_\omega$. Put $\mu_\phi = 1$. Then, assuming $\mu_{\omega_{m-1}^{-1}}$ has been chosen, for all $m \geq 1$ we choose $\mu_{\omega_m^{-1}} \in \mathbb{C}$ of modulus one such that

$$(\lambda_{\omega_{m-1}^{-1}} \mu_{\omega_{m-1}^{-1}}) \overline{\mu_{\omega_m^{-1}}} = c_{\omega_{m-1}^{-1}} \geq 0.$$

Then we have $UT_{i_m}U^*\xi_{\omega_m^{-1}} = c_{\omega_{m-1}^{-1}}\xi_{\omega_{m-1}^{-1}}$ for $m \geq 1$, and this takes care of the main branch scalars. Now, if we are given a word v in \mathbb{F}_ω and μ_v has been chosen, then for $1 \leq i \leq n$ with iv off the main branch, choose scalars μ_{iv} of modulus one such that $(\overline{\mu_v}\lambda_{iv})\mu_{iv} = c_{iv} \geq 0$. Thus, $UT_iU^*\xi_v = c_{iv}\xi_{iv}$ for all v and i , and this yields the desired unitary. ■

The following factorization result will be useful in the sequel.

Proposition 2.5. *Let $T = (T_1, \dots, T_n)$ be a bilateral weighted shift on \mathcal{K}_ω . Then $T_i = S_iW_i$ for $1 \leq i \leq n$, where $S = (S_1, \dots, S_n)$ is the unweighted shift on \mathcal{K}_ω and each W_i is a positive operator, which is diagonal with respect to the standard basis for \mathcal{K}_ω , given by*

$$W_i\xi_u = \lambda_{iu}\xi_u \quad \text{for } 1 \leq i \leq n \quad \text{and } u \in \mathbb{F}_\omega.$$

Furthermore, a subspace \mathcal{H} of \mathcal{K}_ω reduces the family $\{T_1, \dots, T_n\}$, equivalently the algebra $\mathfrak{S}_{\omega, \Lambda}$, if and only if it reduces the family of operators $\{S_1, \dots, S_n, W_1, \dots, W_n\}$.

Proof. The operators T_i are easily seen to factor as $T_i = S_i W_i$, where W_i is given by $W_i \xi_u = (T_i \xi_u, \xi_{iu}) \xi_u = \lambda_{iu} \xi_u$, for $u \in \mathbb{F}_\omega$ and $1 \leq i \leq n$. The last statement follows from standard operator theory. Indeed, if P is a projection which commutes with $T_i = S_i W_i$, then it commutes with $T_i^* T_i = W_i^2$, and hence with the positive operator $W_i = \sqrt{W_i^2}$. Thus, $P S_i W_i = S_i W_i P = S_i P W_i$ and P commutes with S_i on the range of W_i . But by our assumption on weights, W_i is surjective, whence $S_i P = P S_i$. The other direction is obvious. ■

3. MAIN RESULTS

In this section we investigate the reducing subspace structure of the algebras $\mathfrak{S}_{\omega, \Lambda}$ and, at the same time, the joint reducing subspace structure of weighted shifts $T = (T_1, \dots, T_n)$. We begin by addressing the irreducible case.

Proposition 3.1. *Let ω be an aperiodic word in Ω_+ . Then the algebra $\mathfrak{S}_{\omega, \Lambda}$ is irreducible.*

Proof. First note that the projections $\{\omega_m(S)\omega_m(S)^* : m \geq 0\}$ belong to the von Neumann algebra \mathfrak{A}_T generated by $\{T_1, \dots, T_n\}$. This follows from Proposition 2.5, because the von Neumann algebra \mathfrak{A}_S generated by the unweighted shifts $\{S_1, \dots, S_n\}$ is spanned by its projections, and hence the double commutant identity implies $\mathfrak{A}_S = \mathfrak{A}_S'' \subseteq \mathfrak{A}_T'' = \mathfrak{A}_T$.

But when ω is an aperiodic word, the projections $\omega_m(S)\omega_m(S)^*$ converge in the strong operator topology to the rank one projection $\xi_\phi \xi_\phi^*$ onto the subspace $\text{span}\{\xi_\phi\}$. Indeed, the vector ξ_ϕ clearly belongs to the range of the projection $P = \text{sot-lim}_{m \rightarrow \infty} \omega_m(S)\omega_m(S)^*$, which exists since the $\omega_m(S)\omega_m(S)^*$ form a decreasing sequence of projections. On the other hand, given a word $u = v\omega_k^{-1} \in \mathbb{F}_\omega$ and positive integer m , there is a unique word $u_m \in \mathbb{F}_n^+$ of length m such that ξ_u is in the range of $u_m(S)$. This determines an infinite word in Ω_+ , which is $v(\omega_k^{-1}\omega) = u\omega$, and u_m makes up the first m terms of this word. As ω is aperiodic, $u\omega$ and ω are different except when $u = \phi$; for if $u\omega = \omega$, then $|v| \neq k$ implies ω is periodic, whereas $|v| = k$ implies $v = \omega_k$ and $u = \phi$. Hence for some sufficiently large m , the words u_m and ω_m are distinct when $u \neq \phi$. Whence, $\omega_m(S)\omega_m(S)^*\xi_u = 0$, and it follows that these projections converge to $\xi_\phi \xi_\phi^*$.

Thus, it follows that a projection $Q \in \mathfrak{A}'_T$ commuting with the operators $\{T_1, \dots, T_n\}$, also commutes with $P = \xi_\phi \xi_\phi^* \in \mathfrak{A}_T$. But the vector ξ_ϕ is clearly cyclic for \mathfrak{A}_T since all weights are assumed to be nonzero. Hence $Q = 0$ or $Q = I$, as required. \blacksquare

Remark 3.2. We mention that the algebras $\mathfrak{S}_{\omega, \Lambda}$ coming from aperiodic words ω considered in Theorem 3.1 truly are exclusive to the noncommutative setting. Indeed, the only infinite word $\omega = 111\dots$ with letters in a one letter alphabet is, of course, trivially periodic.

We turn now to the periodic case. Given a word $\omega = v_0 v_0 v_0 \dots$ in Ω_+ with $v_0 \in \mathbb{F}_n^+$ primitive (not a power of another element in \mathbb{F}_n^+) and a positive integer $k \geq 1$, there is a canonical decomposition of \mathbb{F}_ω into mutually disjoint subsets, and hence of the Hilbert space $\mathcal{K}_\omega = \ell^2(\mathbb{F}_\omega)$ into mutually orthogonal subspaces. First define for $l \leq 0$

$$\mathbb{F}_{\omega, k}^{(l)} = \{v \omega_m^{-1} : v \in \mathbb{F}_n^+ \setminus \mathbb{F}_n^+ i_m, -lk|v_0| \leq m < (-l+1)k|v_0|\},$$

where we take $i_0 = i_{|v_0|}$. For $l \geq 1$ define

$$\mathbb{F}_{\omega, k}^{(l)} = \{v \omega'_{m+1} : v \in \mathbb{F}_n^+ \setminus \mathbb{F}_n^+ i'_m, (l-1)k|v_0| < m \leq lk|v_0|\},$$

where $\omega' = \dots v_0 v_0$, and $\omega'_m \in \mathbb{F}_n^+$ is the initial segment (from the right) of m letters in this infinite word, with i'_m equal to the m th letter in the sequence. We refer to $\mathbb{F}_{\omega, k}^{(0)} := \mathbb{F}_{\omega, k}$ as the *principal component* of this partition of \mathbb{F}_ω . Observe that the unit ϕ belongs to $\mathbb{F}_{\omega, k}$. Given $u \in \mathbb{F}_{\omega, k}$, further define $\mathcal{J}_{u, k}$ to be the subset of \mathbb{F}_ω consisting of u and all its natural translates in the sets $\{\mathbb{F}_{\omega, k}^{(l)} : l \in \mathbb{Z}\}$. Specifically, define

$$\mathcal{J}_{\phi, k} = \{(v_0^k)^l : l \in \mathbb{Z}\} \quad \text{and} \quad \mathcal{J}_{u, k} = u \mathcal{J}_{\phi, k} \quad \text{for } u \in \mathbb{F}_{\omega, k}.$$

Thus, in summary, given a periodic word $\omega \in \Omega_+$ and a positive integer $k \geq 1$, there is a canonical partition of \mathbb{F}_ω into a disjoint union of subsets which generates a spatial decomposition of \mathcal{K}_ω ;

$$\mathbb{F}_\omega = \bigcup_{u \in \mathbb{F}_{\omega, k}} \mathcal{J}_{u, k} \quad \text{and} \quad \mathcal{K}_\omega = \sum_{u \in \mathbb{F}_{\omega, k}} \oplus \text{span}\{\xi_v : v \in \mathcal{J}_{u, k}\}.$$

Definition 3.3. Let $\omega = v_0 v_0 \dots$ be a periodic word in Ω_+ and let $k \geq 1$ be a positive integer. Let $\mathbb{F}_\omega = \bigcup_{u \in \mathbb{F}_{\omega, k}} \mathcal{J}_{u, k}$ be the associated canonical decomposition of \mathbb{F}_ω . We shall say that a bilateral weighted shift $T = (T_1, \dots, T_n)$ is *period k* (or equivalently, the weights $\Lambda = \{\lambda_u\}_{u \in \mathbb{F}_\omega}$ are period k) if

$$T_i \xi_v = \lambda_{iv} \xi_{iv} = \lambda_u \xi_{iv} \quad \text{for } v \in \mathbb{F}_\omega \quad \text{and} \quad 1 \leq i \leq n,$$

where u is the unique element of $\mathbb{F}_{\omega, k}$ with $iv \in \mathcal{J}_{u, k}$.

Note 3.4. The weights $\{\lambda_u : u \in \mathbb{F}_{\omega,k}\}$ may be regarded as the *remainders* of a k -periodic shift. There is a satisfying visual interpretation of this periodicity in terms of the infinite tree structure of \mathcal{K}_ω . This is expanded on in the examples of the next section. We discovered this notion of periodicity while proving the following theorem.

Theorem 3.5. *The following assertions are equivalent for a periodic word ω in Ω_+ :*

- (i) $\mathfrak{S}_{\omega,\Lambda}$ has non-trivial reducing subspaces.
- (ii) The weights $\Lambda = \{\lambda_u\}_{u \in \mathbb{F}_\omega}$ are periodic.
- (iii) The weighted shift $T = (T_1, \dots, T_n)$ is periodic.

Proof. The last two conditions are equivalent by definition. The implication (ii) \Rightarrow (i) is established in the proof of Theorem 3.10 below. We prove (i) \Rightarrow (ii). Suppose \mathcal{H} is a non-trivial subspace of \mathcal{K}_ω which reduces $\mathfrak{S}_{\omega,\Lambda}$. Let $P_{\mathcal{H}} \notin \{0, I\}$ be the projection onto \mathcal{H} . Then $P_{\mathcal{H}}T_i = T_iP_{\mathcal{H}}$ for $1 \leq i \leq n$, and hence $P_{\mathcal{H}}$ commutes with the family $\{S_i, W_j : 1 \leq i, j \leq n\}$ by Proposition 2.5.

Consider the set \mathcal{P} of all partitions Π of the set \mathbb{F}_ω into mutually disjoint subsets such that

$$Q_{\mathcal{J}}P_{\mathcal{H}} = P_{\mathcal{H}}Q_{\mathcal{J}} \quad \text{for } \mathcal{J} \in \Pi,$$

where $Q_{\mathcal{J}}$ is the projection onto $\text{span}\{\xi_u : u \in \mathcal{J}\}$. Notice that \mathcal{P} is closed under a natural join operation; if $\Pi_1, \Pi_2 \in \mathcal{P}$, then $\Pi_1 \vee \Pi_2 = \{\mathcal{J} : \mathcal{J} = \mathcal{J}_1 \cap \mathcal{J}_2, \mathcal{J}_i \in \Pi_i, i = 1, 2\}$ belongs to \mathcal{P} . This set is clearly a partition of \mathbb{F}_ω into disjoint subsets, and the corresponding projections $Q_{\mathcal{J}} = Q_{\mathcal{J}_1 \cap \mathcal{J}_2} = Q_{\mathcal{J}_1}Q_{\mathcal{J}_2}$ commute with $P_{\mathcal{H}}$.

The following notation will be useful. If we are given $\mathcal{J} \subseteq \mathbb{F}_\omega$ such that $Q_{\mathcal{J}}P_{\mathcal{H}} = P_{\mathcal{H}}Q_{\mathcal{J}}$, then for all words $w \in \mathbb{F}_n^+$ define subsets of \mathbb{F}_ω by

$$w(S)\mathcal{J} \equiv \{wu : u \in \mathcal{J}\} = w\mathcal{J} \quad \text{and} \quad w(S)^*\mathcal{J} \equiv w^{-1}\mathcal{J} \cap \mathbb{F}_\omega,$$

where $w(S)$ is the isometry $w(S) = S_{i_1} \cdots S_{i_k}$ when $w = i_1 \cdots i_k$. Observe that the projections $Q_{w(S)\mathcal{J}}$ and $Q_{w(S)^*\mathcal{J}}$ are, respectively, the projections onto the ranges of the operators $w(S)Q_{\mathcal{J}}$ and $w(S)^*Q_{\mathcal{J}}$. But $w(S)Q_{\mathcal{J}}$ is a partial isometry, so that

$$Q_{w(S)\mathcal{J}} = (w(S)Q_{\mathcal{J}})(w(S)Q_{\mathcal{J}})^* = w(S)Q_{\mathcal{J}}w(S)^*.$$

Moreover, the projection $Q_{\mathcal{J}}$ leaves the range of $w(S)$ invariant since the ranges of both these operators are spanned by standard basis vectors. Thus, $w(S)^*Q_{\mathcal{J}}$ is also a partial isometry with final projection

$$\begin{aligned} Q_{w(S)^*\mathcal{J}} &= (w(S)^*Q_{\mathcal{J}}w(S))(w(S)^*Q_{\mathcal{J}}w(S))^* \\ &= w(S)^*Q_{\mathcal{J}}(w(S)w(S)^*)Q_{\mathcal{J}}w(S) = w(S)^*Q_{\mathcal{J}}w(S). \end{aligned}$$

Hence these projections commute with $P_{\mathcal{H}}$ as well. Further define subsets $v(S)w(S)^*\mathcal{J}$, with $v, w \in \mathbb{F}_n^+$, in a similar manner, yielding projections $Q_{v(S)w(S)^*\mathcal{J}} = v(S)w(S)^*Q_{\mathcal{J}}w(S)v(S)^*$ that commute with $P_{\mathcal{H}}$ as well.

Next define a distinguished partition Π_0 in \mathcal{P} , determined by the equivalence relation

$$u \sim v \quad \text{for } u, v \in \mathbb{F}_{\omega} \quad \text{if and only if } \lambda_u = \lambda_v.$$

To see that Π_0 belongs to \mathcal{P} , for $r > 0$ let Q_r be the projection onto $\text{span}\{\xi_u : \lambda_u = r\}$. If nonzero, this is a typical projection onto a subspace of \mathcal{K}_{ω} determined by a coset of Π_0 . For $1 \leq i \leq n$, let $P_{r,i}$ be the projection onto the eigenspace

$$P_{r,i}\mathcal{K}_{\omega} = \ker(W_i - rI) = \text{span}\{\xi_u : \lambda_{iu} = r\}.$$

As spectral projections for the W_i , the $P_{r,i}$ commute with $P_{\mathcal{H}}$ and so do the operators $S_i P_{r,i}$. Let $Q_{r,i} = S_i P_{r,i} S_i^*$ be the projection onto the range of the partial isometry $S_i P_{r,i}$, which is $\text{span}\{\xi_v : \lambda_v = r, v = iu, u \in \mathbb{F}_{\omega}\}$. Then $Q_r = \sum_i Q_{r,i}$ is a projection which evidently commutes with $P_{\mathcal{H}}$. Thus Π_0 belongs to \mathcal{P} as claimed.

We define a partial ordering on the partitions \mathcal{P} by: $\Pi_1 \geq \Pi_2$ if every $\mathcal{J}_2 \in \Pi_2$ may be obtained by $\mathcal{J}_2 = \bigcup_{\mathcal{J}_1 \subseteq \mathcal{J}_2} \mathcal{J}_1$ with $\mathcal{J}_1 \in \Pi_1$. Let $\mathcal{P}_0 = \{\Pi \in \mathcal{P} : \Pi \geq \Pi_0\}$. We claim that every chain $\{\Pi_{\alpha} : \alpha \in \mathbb{A}\}$ in \mathcal{P}_0 has a maximal element Π' in \mathcal{P}_0 . Indeed, define the partition Π' by the equivalence relation $u \sim v$, for $u, v \in \mathbb{F}_{\omega}$, if for any $\alpha \in \mathbb{A}$, there exists $\mathcal{J}_{\alpha} \in \Pi_{\alpha}$ such that $u, v \in \mathcal{J}_{\alpha}$. This is an equivalence relation since each Π_{α} is a partition of \mathbb{F}_{ω} into disjoint subsets. As the Π_{α} form a chain, it is clear that $\Pi' \geq \Pi_{\alpha} \geq \Pi_0$ for all $\alpha \in \mathbb{A}$. Let $\mathcal{J} \in \Pi'$. Observe from the definition of this relation, that for all $\alpha \in \mathbb{A}$, there is a unique $\mathcal{J}_{\alpha} \in \Pi_{\alpha}$ with $\mathcal{J} \subseteq \mathcal{J}_{\alpha}$. On the other hand, if $u \in \cap_{\alpha \in \mathbb{A}} \mathcal{J}_{\alpha} \supseteq \mathcal{J}$, then $u \sim v$ for all $v \in \mathcal{J}$. Hence $u \in \mathcal{J}$, and we have $\mathcal{J} = \cap_{\alpha \in \mathbb{A}} \mathcal{J}_{\alpha}$. Further, we have $\mathcal{J}_{\alpha} \supseteq \mathcal{J}_{\beta} \supseteq \mathcal{J}$ for $\Pi_{\alpha} \leq \Pi_{\beta}$ by the definition of the ordering. Thus, it follows that $P_{\mathcal{H}}$ commutes with the projection $Q_{\mathcal{J}} = Q_{\cap_{\alpha} \mathcal{J}_{\alpha}} = \bigwedge_{\alpha} Q_{\mathcal{J}_{\alpha}}$. Hence Π' belongs to \mathcal{P}_0 and majorizes the chain $\{\Pi_{\alpha} : \alpha \in \mathbb{A}\}$ as claimed.

Now apply Zorn's Lemma to obtain a maximal element $\overline{\Pi}$ of \mathcal{P}_0 . We set aside a pair of technical results on $\overline{\Pi}$ which we require:

Lemma 3.6. *Let $\mathcal{J} \in \overline{\Pi}$. For $m \geq 0$, there is a unique word $w \in \mathbb{F}_n^+$, $|w| = m$, such that $w(S)^*\mathcal{J} \neq \emptyset$. It follows that the projections $\{Q_{v(S)w(S)^*\mathcal{J}} : v \in \mathbb{F}_n^+\}$ are minimal projections with ranges spanned by standard basis vectors in the commutant $\{P_{\mathcal{H}}\}'$. In particular, if $\mathcal{J}_1 \in \overline{\Pi}$ is such that $v(S)w(S)^*\mathcal{J} \cap \mathcal{J}_1 \neq \emptyset$, then $v(S)w(S)^*\mathcal{J} = \mathcal{J}_1$.*

Proof. Given $m \geq 0$, there is some word $w \in \mathbb{F}_n^+$, $|w| = m$, for which $w(S)^* \mathcal{J} \neq \emptyset$ since $\mathcal{J} \neq \emptyset$. As $w(S)w(S)^* \mathcal{J} \neq \emptyset$, we in fact have $w(S)w(S)^* \mathcal{J} = \mathcal{J}$. Indeed, we clearly have $w(S)w(S)^* \mathcal{J} \subseteq \mathcal{J}$, and if this were a strict inclusion we could refine $\overline{\Pi}$ in \mathcal{P}_0 by replacing \mathcal{J} by $w(S)w(S)^* \mathcal{J}$ and $\mathcal{J} \setminus w(S)w(S)^* \mathcal{J}$. It follows that there can be no other word $w' \in \mathbb{F}_n^+$, $|w'| = m$, with $w'(S)^* \mathcal{J} \neq \emptyset$.

Suppose $Q \leq Q_{v(S)w(S)^* \mathcal{J}} = v(S)w(S)^* Q_{\mathcal{J}} w(S)v(S)^*$ is a nonzero projection in $\{P_{\mathcal{H}}\}'$ with range spanned by standard basis vectors. Then it follows that $w(S)v(S)^* Q v(S)w(S)^*$ is a nonzero projection in $\{P_{\mathcal{H}}\}'$, whose range is spanned by standard basis vectors, with

$$\begin{aligned} w(S)v(S)^* Q v(S)w(S)^* &\leq w(S)w(S)^* Q_{\mathcal{J}} w(S)v(S)^* \\ &= Q_{w(S)w(S)^* \mathcal{J}} = Q_{\mathcal{J}}. \end{aligned}$$

Hence $w(S)v(S)^* Q v(S)w(S)^* = Q_{\mathcal{J}}$ by the minimality of $Q_{\mathcal{J}}$ as a projection spanned by standard basis vectors in $\{P_{\mathcal{H}}\}'$ (which follows from the maximality of the partition $\overline{\Pi}$). Thus it follows that

$$\begin{aligned} Q &= v(S)v(S)^* Q v(S)v(S)^* \\ &= v(S)w(S)^* (w(S)v(S)^* Q v(S)w(S)^*) w(S)v(S)^* \\ &= Q_{v(S)w(S)^* \mathcal{J}}. \end{aligned}$$

Finally, let $\mathcal{J}_1 \in \overline{\Pi}$ be such that $v(S)w(S)^* \mathcal{J} \cap \mathcal{J}_1 \neq \emptyset$. Then $\mathcal{J}_1 \subseteq v(S)w(S)^* \mathcal{J}$; otherwise we could refine $\overline{\Pi}$ by replacing \mathcal{J}_1 by $\mathcal{J}_1 \cap v(S)w(S)^* \mathcal{J}$ and $\mathcal{J}_1 \setminus v(S)w(S)^* \mathcal{J}$. But by the previous minimality argument we in fact have $\mathcal{J}_1 = v(S)w(S)^* \mathcal{J}$, as required. ■

Lemma 3.7. *Every $\mathcal{J} \in \overline{\Pi}$ contains more than one element. (In fact, we will see that every $\mathcal{J} \in \overline{\Pi}$ contains infinitely many elements.) Moreover, there is a $\mathcal{J}_1 \in \overline{\Pi}$ with two elements lying on the same path in \mathbb{F}_{ω} , and it follows that there is a word $w \in \mathbb{F}_n^+$ such that $w(S)\mathcal{J}_1 = \mathcal{J}_1$. In particular, \mathcal{J}_1 contains elements lying on the main branch of \mathbb{F}_{ω} .*

Proof. Let $\mathcal{J} \in \overline{\Pi}$. Then \mathcal{J} is not a singleton. Suppose to the contrary that $\mathcal{J} = \{u\}$. For all $v \in \mathbb{F}_{\omega}$, there are words v_+, v_- in \mathbb{F}_n^+ such that $v_+(S)v_-(S)^* \xi_u = \xi_v$. In particular, the nonempty subsets amongst $\{v_+(S)v_-(S)^* \mathcal{J}' : \mathcal{J}' \in \overline{\Pi}\}$ are part of a partition in \mathcal{P} which includes the subset $\{v\}$. Since $v \in \mathbb{F}_{\omega}$ was arbitrary, and \mathcal{P} is closed under the meet operation, it follows that the partition $\{\{v\} : v \in \mathbb{F}_{\omega}\}$ belongs to \mathcal{P} . Hence each vector ξ_v either belongs to \mathcal{H} or is orthogonal to it. Thus, as $\mathcal{H} \neq \{0\}$, it includes some standard basis vector ξ_{v_0} . However, every standard basis vector is cyclic for the von Neumann algebra \mathfrak{A}_S , and $P_{\mathcal{H}}$ belongs to the commutant \mathfrak{A}'_S . This implies, incorrectly, that $\mathcal{H} = \mathcal{K}_{\omega}$ since $\mathcal{H} \supseteq \overline{\mathfrak{A}_S \xi_{v_0}} = \mathcal{K}_{\omega}$. Thus \mathcal{J} contains at least two elements.

Now suppose $u_1 \neq u_2$ belong to \mathcal{J} with $|u_2| \geq |u_1|$. Let $u \in \mathbb{F}_\omega$ be the nearest common ancestor of u_1, u_2 . Then u lies closer to u_1 in the tree structure for \mathcal{K}_ω . Suppose $u \in \mathcal{J}_1 \in \overline{\Pi}$. Choose $v_1 \in \mathbb{F}_n^+$ such that $\xi_{u_1} = v_1(S)\xi_u$. Then $v_1(S)\mathcal{J}_1 \cap \mathcal{J} \neq \emptyset$, and hence we have $\mathcal{J} = v_1(S)\mathcal{J}_1$ by Lemma 3.6. Thus $\xi_{u_2} = v_1(S)\xi_x$ for some $x \in \mathcal{J}_1$. Since $|u_2| \geq |u_1|$, we have $|x| \geq |u|$, and the existence of a $w \in \mathbb{F}_n^+$ such that $w(S)\xi_u = \xi_x$ follows from u and x both being ancestors of u_2 . As $u = x$ would incorrectly imply $\xi_{u_1} = \xi_{u_2}$, we have shown that \mathcal{J}_1 contains elements $u \neq x$ for which $w(S)\xi_u = \xi_x$ for some $w \in \mathbb{F}_n^+$. But another application of Lemma 3.6 yields $w(S)\mathcal{J}_1 = \mathcal{J}_1$, since these sets have non-trivial intersection. Thus $(w(S)^*)^m \mathcal{J}_1 = \mathcal{J}_1$ for $m \geq 0$. It follows that \mathcal{J}_1 contains a pair of elements (infinitely many in fact) which lie on the main branch. \blacksquare

Let $\mathcal{J}_1 \in \overline{\Pi}$ be obtained as in Lemma 3.7, and let $u \in \mathbb{F}_n^+$ be a word of minimal length such that $u(S)\mathcal{J}_1 = \mathcal{J}_1$. We may assume with no loss of generality that the unit $\phi \in \mathcal{J}_1$. Indeed, there is a $v \in \mathbb{F}_n^+$ with $\phi \in v(S)\mathcal{J}_1 \in \overline{\Pi}$, and evidently $v(S)\mathcal{J}_1$ will satisfy the conditions on \mathcal{J}_1 in Lemma 3.7. Thus the vectors $(u(S)^*)^m \xi_\phi$, $m \geq 0$, are standard basis vectors on the main branch. It follows that if $\omega = v_0 v_0 v_0 \dots$ with $v_0 \in \mathbb{F}_n^+$ a primitive word, then there exists positive integers m_1, m_2 for which $v_0^{m_1} = u^{m_2}$. Thus we can apply the following combinatorial lemma:

Lemma 3.8. *Suppose $u, v \in \mathbb{F}_n^+$ and $l, m \geq 1$ are positive integers such that $u^l = v^m$. If v is a primitive word, then $u = v^k$ for some $k \geq 1$.*

Proof. Suppose first that $|v| \leq |u|$. We can clearly assume $l, m >> 0$. Then $u^l = v^m$ implies the existence of $u_1, u_2 \in \mathbb{F}_n^+$ such that $u = vu_1$ and $u = u_2v$ with $|u_1| = |u_2|$. If $u_1, u_2 \neq \phi$ (or $u \neq v$), then $u_2 = vu_3$, $u_1 = u_4v$, but $vu_3v = u = vu_4v$, whence $u_3 = u_4$. If $u_3 \neq \phi$ (or $u \neq v^2$), then $u_3 = vu_5 = u_6v$ with $|u_5| = |u_6|$. If $u_5, u_6 \neq \phi$ (or $u \neq v^3$), then $u_5 = u_7v$, $u_6 = vu_8$ and $u_7 = u_8$. We can iterate this process just finitely many times, hence we eventually get $u = v^k$ for some $k \geq 1$, as required. On the other hand, by the previous argument the $|u| \leq |v|$ case forces $u = v$ by primitivity of v . \blacksquare

Therefore, we have $u = v_0^k$ for some $k \geq 1$, and it follows that

$$\mathcal{J}_1 = \{(v_0^k)^l : l \in \mathbb{Z}\} \equiv \mathcal{J}_{\phi, k}.$$

Indeed, this set is contained in \mathcal{J}_1 as $\phi \in \mathcal{J}_1 = u(S)^l \mathcal{J}_1 = (u(S)^*)^l \mathcal{J}_1$, for $l \geq 0$. Also, the minimality of the length of u such that $u(S)\mathcal{J}_1 = \mathcal{J}_1$ ensures that the elements $\{(v_0^k)^l : l \leq 0\}$ form the intersection of \mathcal{J}_1 and the main branch. Thus if $v \in \mathcal{J}_1$, then some power $(u(S)^*)^l \xi_v = \xi_{u^{-l}v}$

will reside on the main branch, forcing $u^{-l}v$, and hence $v = u^l(u^{-l}v)$ to belong to this set.

We have just shown that the set \mathcal{J}_1 coincides with the set $\mathcal{J}_{\phi,k}$ from the discussion prior to Definition 3.3. Thus, it follows from our analysis of the partition $\overline{\Pi}$ that

$$\overline{\Pi} = \{u\mathcal{J}_{\phi,k} : u \in \mathbb{F}_{\omega,k}\} = \{\mathcal{J}_{u,k} : u \in \mathbb{F}_{\omega,k}\}.$$

Therefore, the periodicity of $\mathfrak{S}_{\omega,\Lambda}$ and $T = (T_1, \dots, T_n)$ is now an immediate consequence of $\overline{\Pi} \geq \Pi_0$. This completes the proof of Theorem 3.5. \blacksquare

Remark 3.9. This result generalizes Nikolskii's Theorem for the single variable case (Theorem 4 from [28]), which asserts that bilateral weighted shift operators on Hilbert space have non-trivial reducing subspaces if and only if the associated weight sequence is periodic. In fact, the reader of [28] will notice that we have been inspired by the proof of this theorem in establishing the implication $(i) \Rightarrow (ii)$ of Theorem 3.5. However, we wish to emphasize that, evidently, there are a number of technical details here which do not arise in the single variable case.

We now combine the previous results and show how, in the doubly-periodic case, non-trivial reducing subspaces are determined by reducing subspaces of the free semigroup algebra \mathfrak{S}_ω . We require another decomposition of \mathbb{F}_ω when ω is periodic (see the discussion prior to Definition 3.3). For $l \in \mathbb{Z}$, partition $\mathbb{F}_{\omega,k}^{(l)}$ into k disjoint subsets $\mathbb{F}_{\omega,k}^{(l)} = \mathbb{G}_0^{(l)} \cup \dots \cup \mathbb{G}_{k-1}^{(l)}$. To avoid cumbersome notation, we only explicitly define the decomposition of the principal component $\mathbb{F}_{\omega,k} \equiv \mathbb{F}_{\omega,k}^{(0)} = \mathbb{G}_0^{(0)} \cup \dots \cup \mathbb{G}_{k-1}^{(0)}$, the other cases are similar. Let

$$\mathbb{G}_r^{(0)} = \{v\omega_m^{-1} : v \in \mathbb{F}_n^+, r|v_0| \leq m < (r+1)|v_0|\} \quad \text{for } 0 \leq r < k.$$

Let $\mathcal{K}_{\omega,k}$ be the subspace of \mathcal{K}_ω defined by

$$\mathcal{K}_{\omega,k} := \sum_{m \in \mathbb{Z}} \oplus \text{span}\{\xi_u : u \in \mathbb{G}_0^{(mk)}\}.$$

Given $m \in \mathbb{Z}$, there is a natural bijection between the sets $\mathbb{G}_0^{(mk)}$ and $\mathbb{G}_0^{(m)}$ which determines a unitary operator $U_{\omega,k} : \mathcal{K}_{\omega,k} \rightarrow \mathcal{K}_\omega$ mapping $\mathcal{K}_{\omega,k}$ onto $\mathcal{K}_\omega = \sum_{m \in \mathbb{Z}} \oplus \text{span}\{\xi_u : u \in \mathbb{G}_0^{(m)}\}$.

Theorem 3.10. *The following assertions are equivalent for an infinite word ω in Ω_+ :*

- (i) *The algebra $\mathfrak{S}_{\omega,\Lambda}$ has non-trivial reducing subspaces.*

- (ii) The operators $T = (T_1, \dots, T_n)$ have non-trivial joint reducing subspaces.
- (iii) The pair (ω, Λ) is doubly-periodic; that is, the word ω is periodic and the weights $\Lambda = \{\lambda_u\}_{u \in \mathbb{F}_\omega}$ are periodic.

Furthermore, in the case that the word $\omega = v_0v_0 \cdots$ is periodic and the weights $\Lambda = \{\lambda_u\}_{u \in \mathbb{F}_\omega}$ are of period k , the reducing subspaces $\mathcal{H} \subseteq \mathcal{K}_\omega$ for $\mathfrak{S}_{\omega, \Lambda}$, equivalently the joint reducing subspaces for $T = (T_1, \dots, T_n)$, are generated by the reducing subspaces for the free semigroup algebra \mathfrak{S}_ω in the following manner:

$$(3) \quad \mathcal{H} = \sum_{u \in \mathbb{F}_{\omega, k}} \oplus Q_{\mathcal{J}_{u, k}} \mathcal{H} = \sum_{r=0}^{k-1} \oplus P_r \mathcal{H},$$

where the projections

$$(4) \quad P_r = \sum_{u \in \mathbb{G}_r^{(0)}} \oplus Q_{\mathcal{J}_{u, k}} = \sum_{u \in \mathbb{G}_r^{(0)}} \oplus u(S)Q_{\mathcal{J}_{\phi, k}}u(S)^*,$$

for $0 \leq r < k$, commute with $P_\mathcal{H}$. The subspace $P_0 \mathcal{H}$ is contained in $\mathcal{K}_{\omega, k}$, and the image $U_{\omega, k} P_0 \mathcal{H}$ is a reducing subspace for the free semigroup algebra \mathfrak{S}_ω . Conversely, every reducing subspace \mathcal{H}' for the free semigroup algebra \mathfrak{S}_ω generates a reducing subspace \mathcal{H} for $\mathfrak{S}_{\omega, \Lambda}$ in this way through the equation $P_0 \mathcal{H} \equiv U_{\omega, k}^* \mathcal{H}'$.

Proof. The only thing left to establish is the form of the reducing subspaces for $\mathfrak{S}_{\omega, \Lambda}$ when both ω and Λ are periodic. The form of \mathcal{H} in (3) follows from the proof of Theorem 3.5. In that proof, we saw how the projections $Q_{\mathcal{J}_{u, k}}$ satisfy $Q_{\mathcal{J}_{u, k}} = Q_{u \mathcal{J}_{\phi, k}} = u(S)Q_{\mathcal{J}_{\phi, k}}u(S)^*$. The fact that $\mathcal{H}' = U_{\omega, k} P_0 \mathcal{H}$ reduces \mathfrak{S}_ω , equivalently S_1, \dots, S_n , follows from; the k -periodicity of the weights Λ , equations (3) and (4), and the projections $Q_{\mathcal{J}_{u, k}}$ commuting with $P_\mathcal{H}$. We leave this technical detail to the interested reader. On the other hand, every reducing subspace \mathcal{H}' for \mathfrak{S}_ω can be seen to generate an $\mathfrak{S}_{\omega, \Lambda}$ -reducing subspace by first defining $P_0 \mathcal{H} \equiv U_{\omega, k}^* \mathcal{H}'$, and then obtaining the rest of the subspaces $P_r \mathcal{H}$ by translations. ■

Remark 3.11. For the sake of brevity, we have deliberately avoided some technical details in giving the characterization of reducing subspaces for $\mathfrak{S}_{\omega, \Lambda}$ in the doubly-periodic case of Theorem 3.10. Suffice it to say, the reducing subspaces for $\mathfrak{S}_{\omega, \Lambda}$ are precisely those subspaces which are generated in a natural way by reducing subspaces for the unweighted shifts, in a manner which is analogous to the single variable case $n = 1$ [28]. We mention that reducing subspaces for the unweighted shifts $S = (S_1, \dots, S_n)$, equivalently for \mathfrak{S}_ω , were discussed

by Davidson and Pitts [15] in the context of representation theory for free semigroup algebras, hence we shall not pursue this here.

4. PERIODIC EXAMPLES

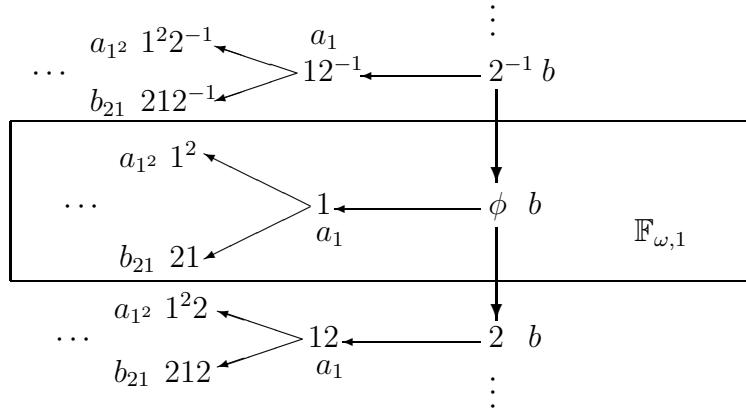
In this section, we explore further the notion of periodicity discovered in Theorem 3.5. In particular, by considering the simplest possible examples, we show there are natural infinite ‘shift matrices’ associated with periodic n -tuples $T = (T_1, \dots, T_n)$. This gives information on the various operator algebras generated by these shifts, and leads us to a number of open problems posed at the end of this section.

Example 4.1. Let $n = 2$ and let $\omega = 222\dots$ belong to Ω_+ . We shall consider weighted shifts $T = (T_1, T_2)$ acting on \mathcal{K}_ω of period $k = 1$ and $k = 2$. Unlike the single variable case, there is diversity even amongst the class of 1-periodic shifts.

(i) Let $T = (T_1, T_2)$ be a weighted shift on \mathcal{K}_ω of period $k = 1$. The 1-periodicity of T implies the actions of T_1, T_2 are completely determined by their behaviour on the principal component $\mathbb{F}_{\omega,1}$, which has the following natural lexicographic ordering,

$$\begin{aligned}\mathbb{F}_{\omega,1} &= \{\{\phi\}, \{v \in \mathbb{F}_n^+ 1\}\} \\ &= \{\{\phi\}, \{1\}, \{1^2, 21\}, \{1^3, 121, 21^2, 2^21\}, \dots\}.\end{aligned}$$

The Hilbert space \mathcal{K}_ω has an infinite tree structure traced out by the operators T_1, T_2 . Vertices are identified with standard basis vectors $\{\xi_u : u \in \mathbb{F}_\omega\}$ and weights $\Lambda = \{\lambda_u\}_{u \in \mathbb{F}_\omega}$. The tree structure is as follows in this case:



As an illustration of how $T = (T_1, T_2)$ acts on basis vectors, we note

$$\begin{cases} T_1 \xi_\phi = a_1 \xi_1 \\ T_2 \xi_\phi = b \xi_2 \end{cases} \quad \text{and} \quad \begin{cases} T_1 \xi_1 = a_{1^2} \xi_{1^2} \\ T_2 \xi_1 = b_{21} \xi_{21} \end{cases}$$

With our previous notation, the set $\mathcal{J}_{\phi,1}$ here consists of all the main branch elements together with the positive powers, $\mathcal{J}_{\phi,1} = \{2^l : l \in \mathbb{Z}\}$, and $\mathcal{J}_{u,1} = u\mathcal{J}_{\phi,1}$ for $u \in \mathbb{F}_{\omega,1}$. Consider the subspaces

$$\mathcal{H}_u = \text{span}\{\xi_v : v \in \mathcal{J}_{u,1}\} = u(S)\mathcal{H}_\phi \quad \text{for } u \in \mathbb{F}_{\omega,1}.$$

Observe that $\mathcal{K}_\omega = \sum_{u \in \mathbb{F}_{\omega,1}}^\oplus \mathcal{H}_u$ since \mathbb{F}_ω is partitioned by $\{\mathcal{J}_{u,1} : u \in \mathbb{F}_{\omega,1}\}$, and the adjoint operator $u(S)^*$ acts as a unitary from $\mathcal{H}_u = u(S)\mathcal{H}_\phi$ onto \mathcal{H}_ϕ . Thus, putting these equivalences together yields $\mathcal{K}_\omega \simeq \sum_{u \in \mathbb{F}_{\omega,1}}^\oplus \mathcal{H}_\phi$, and if we order the index set $\mathbb{F}_{\omega,1}$ as above, then we obtain the following block matrix decompositions for T_1 and T_2 (up to unitary equivalence):

$$T_1 \simeq \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ a_1 I & 0 & 0 & 0 \\ \hline 0 & a_{1^2} I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & a_{1^3} I & 0 \\ 0 & 0 & 0 & a_{121} I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline & & \ddots & \\ & & & \ddots \\ & & & & \ddots \end{array} \right]$$

and

$$T_2 \simeq \left[\begin{array}{cc|cc} b U & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & b_{21} I & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b_{21^2} I & 0 \\ 0 & 0 & 0 & b_{221} I \\ \hline & & \ddots & \\ & & & \ddots \\ & & & & \ddots \end{array} \right]$$

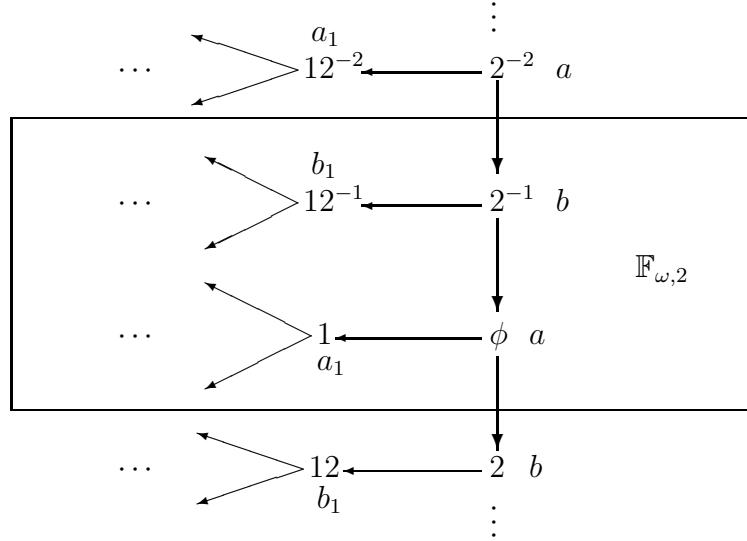
where U is the canonical (unweighted) bilateral weighted shift operator acting on the standard basis for \mathcal{H}_ϕ .

These decompositions can yield information on operator algebras generated by the T_i . For instance, if we let $\mathfrak{A}_{\omega,1}$ be the C^* -algebra (contained in $\mathcal{B}(\mathcal{K}_\omega)$) generated by the T_1, T_2 from all 1-periodic shifts acting on \mathcal{K}_ω , then we have complete freedom on choices of scalars a_u, b_v determining the generators of $\mathfrak{A}_{\omega,1}$. Hence, it is evident from these decompositions that $\mathfrak{A}_{\omega,1}$ is unitarily equivalent to an algebra which is $*$ -isomorphic to $\mathcal{B}(\mathcal{H}) \otimes C(\mathbb{T})$, where \mathcal{H} is a separable infinite dimensional Hilbert space and $C(\mathbb{T})$ is the set of continuous functions on the unit circle (which is isomorphic to $C^*(U)$).

(ii) Let $T = (T_1, T_2)$ be a weighted shift on \mathcal{K}_ω of period $k = 2$. We may obtain similar block matrix decompositions for T_1 and T_2 . In this case, the actions of T_1, T_2 depend on their behaviour on the principal component $\mathbb{F}_{\omega,2}$, which has the ordering

$$\begin{aligned}\mathbb{F}_{\omega,2} &= \{\{\phi\}, \{v \in \mathbb{F}_n^+ 1\}, \{2^{-1}\}, \{v \in \mathbb{F}_n^+ 12^{-1}\}\} \\ &= \left\{ \begin{array}{l} \{\phi\}, \{1\}, \{v1 : v \in \mathbb{F}_n^+, |v| = 1\}, \dots, \\ \{2^{-1}\}, \{12^{-1}\}, \{v12^{-1} : v \in \mathbb{F}_n^+, |v| = 1\}, \dots \end{array} \right\}.\end{aligned}$$

The following diagram gives the infinite tree structure here:



From Theorem 3.10, we have $\mathcal{K}_\omega = P_0\mathcal{K}_\omega \oplus P_1\mathcal{K}_\omega$, and it is easy to see that $P_0\mathcal{K}_\omega$ and $P_1\mathcal{K}_\omega$ are reducing subspaces for T_1 in this example. Further, it is clear from the matrix decomposition associated with T_1 ,

that the restrictions $T_1|_{P_0\mathcal{K}_\omega}$ and $T_1|_{P_1\mathcal{K}_\omega}$ are each unitarily equivalent to operators in the T_1 class of the 1-periodic shifts above. Thus T_1 is unitarily equivalent to the orthogonal direct sum of two operators $T_1 \simeq T_1^{(0)} \oplus T_1^{(1)}$ from the 1-periodic case. On the other hand, when we use the above ordering on $\mathbb{F}_{\omega,2}$, and make the spatial identifications $\mathcal{H}_u \simeq \mathcal{H}_\phi$ as in the above example (where these are the subspaces obtained in the 2-periodic case), we decompose

$$\mathcal{K}_\omega = P_0\mathcal{K}_\omega \oplus P_1\mathcal{K}_\omega \simeq \left(\sum_{u \in \mathbb{G}_0^{(0)}} \oplus \mathcal{H}_\phi \right) \oplus \left(\sum_{u \in \mathbb{G}_1^{(0)}} \oplus \mathcal{H}_\phi \right).$$

For $u, v \in \mathbb{F}_{\omega,2}$, let E_{uv} be the matrix unit corresponding to the (u, v) entry in this decomposition of $\mathcal{B}(\mathcal{K}_\omega)$. Then we may write the block matrix decomposition unitarily equivalent to T_2 as:

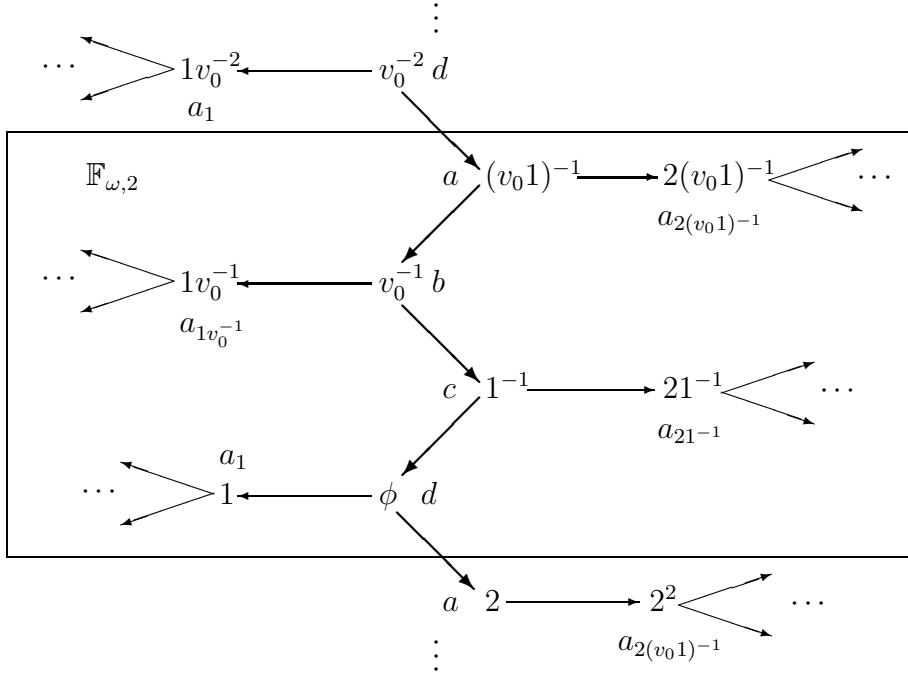
$$\begin{aligned} T_2 &\simeq (bU)E_{2^{-1},\phi} + \sum_{u \in \mathbb{G}_0^{(0)}; u \neq \phi} a_{2u}E_{2u,u} \\ &\quad + aE_{\phi,2^{-1}} + \sum_{u \in \mathbb{G}_1^{(0)}; u \neq 2^{-1}} b_{2u}E_{2u,u}, \end{aligned}$$

with U the bilateral shift acting on the standard basis for \mathcal{H}_ϕ , which is $\{\xi_{v_l} : v_l = 2^{2l}, l \in \mathbb{Z}\}$ in this case. Again, we may deduce properties of the operator algebras generated by such weighted shifts. For instance, it follows from these matrix decompositions that the C^* -algebra $\mathfrak{A}_{\omega,2}$ is $*$ -isomorphic to $\mathcal{B}(\mathcal{H}) \otimes C(\mathbb{T})$ as in the 1-periodic case. In fact, it can be shown that all the C^* -algebras $\mathfrak{A}_{\omega,k}$, $k \geq 1$, are $*$ -isomorphic. We address this point below.

Example 4.2. Let $n = 2$ and let $\omega = v_0v_0v_0\cdots$ belong to Ω_+ with $v_0 = 12$. Consider the weighted shifts $T = (T_1, T_2)$ acting on \mathcal{K}_ω of period $k = 2$. The principal component, with its natural ordering, is given by

$$\mathbb{F}_{\omega,2} = \left\{ \begin{array}{l} \{\phi\}, \{v \in \mathbb{F}_n^+ 1\}, \{1^{-1}\}, \{v \in \mathbb{F}_n^+ 21^{-1}\}, \\ \{v_0^{-1}\}, \{v \in \mathbb{F}_n^+ 1v_0^{-1}\}, \{(v_0 1)^{-1}\}, \{v \in \mathbb{F}_n^+ 2(v_0 1)^{-1}\} \end{array} \right\}.$$

The infinite tree structure for \mathcal{K}_ω traced out by a given 2-periodic shift $T = (T_1, T_2)$ is:



In this case, $\mathcal{J}_{\phi,2} = \{v_0^{2l} : l \in \mathbb{Z}\}$ and $\mathcal{J}_{u,2} = u\mathcal{J}_{\phi,2}$ for $u \in \mathbb{F}_{\omega,2}$. Thus $\mathcal{K}_\omega = \sum_{u \in \mathbb{F}_{\omega,2}} \oplus \mathcal{H}_u \simeq \sum_{u \in \mathbb{F}_{\omega,2}} \oplus \mathcal{H}_\phi$, and with the above ordering on $\mathbb{F}_{\omega,2}$, we have the corresponding block matrix decompositions for T_1 and T_2 :

$$T_1 \simeq b E_{v_0^{-1},(v_01)^{-1}} + d E_{\phi,1^{-1}} + \sum_{u \in \mathbb{F}_{\omega,2}; u \notin \{1^{-1}, (v_01)^{-1}\}} a_{1u} E_{1u,u},$$

and

$$T_2 \simeq (a U) E_{(v_01)^{-1},\phi} + c E_{1^{-1},v_0^{-1}} + \sum_{u \in \mathbb{F}_{\omega,2}; u \notin \{\phi, v_0^{-1}\}} a_{2u} E_{2u,u}.$$

where again U is the canonical bilateral weighted shift operator on the standard basis for \mathcal{H}_ϕ , which is $\{\xi_{v_l} : v_l = v_0^{2l}, l \in \mathbb{Z}\}$.

From discussions in the previous examples, we may deduce the following result. Given a periodic word ω in Ω_+ and a positive integer $k \geq 1$, let $\mathfrak{A}_{\omega,k}$ be the C^* -algebra (contained in $\mathcal{B}(\mathcal{K}_\omega)$) generated by all the T_i from every k -periodic shift $T = (T_1, \dots, T_n)$ acting on \mathcal{K}_ω .

Theorem 4.3. *Let $n \geq 2$. For every periodic word ω in Ω_+ and positive integer $k \geq 1$, $\mathfrak{A}_{\omega,k}$ is $*$ -isomorphic to $\mathcal{B}(\mathcal{H}) \otimes C(\mathbb{T})$, where \mathcal{H} is a separable infinite dimensional Hilbert space.*

Proof. It is evident from the previous examples that $\mathfrak{A}_{\omega,k}$ is unitarily equivalent to a C^* -subalgebra of $\mathfrak{A} = \mathcal{B}(\mathcal{H}) \otimes C(\mathbb{T})$. Furthermore, each matrix unit E_{uv} , for $u, v \in \mathbb{F}_{\omega,k}$, is present in this subalgebra, and the general matrix decomposition for exactly one of the T_i has a weight multiple of the canonical bilateral shift in one of its entries. As there is complete freedom on choices of weights for the generators of $\mathfrak{A}_{\omega,k}$, it follows that this subalgebra is in fact the entire algebra \mathfrak{A} . ■

Remark 4.4. This result illustrates a difference between the commutative ($n = 1$) and noncommutative ($n \geq 2$) cases. Indeed, for $n = 1$, the C^* -algebra generated by all k -periodic bilateral weighted shift operators, with respect to a given basis, is easily seen to be isomorphic to the algebra $M_k(C(\mathbb{T}))$ of $k \times k$ matrices with entries in $C(\mathbb{T})$. (These algebras played a role in the work of Bunce and Deddens [8, 9].) Whereas for $n \geq 2$, the connection with k is washed away by the infinite multiplicities present, at least in this C^* -algebra setting. The nonselfadjoint versions of $\mathfrak{A}_{\omega,k}$ will clearly be unitarily equivalent to matrix function algebras as well; however, we would expect distinct algebras for different values of k in the nonselfadjoint case.

4.1. Concluding Remarks and Open Problems. There are a number of open problems on the operator algebras determined by these weighted shifts. For instance, we have not addressed the classification problem for the algebras $\mathfrak{S}_{\omega,\Lambda}$. It would be interesting to know if $\mathfrak{S}_{\omega,\Lambda}$ can be classified by spatial means strictly in terms of ω and Λ , up to some sort of shift-tail equivalence classes.

Problem 4.5. Does the pair (ω, Λ) form a complete set of unitary invariants for $\mathfrak{S}_{\omega,\Lambda}$?

More generally, we wonder about the representation theory for these algebras [20], as well as reflexivity issues.

Problem 4.6. When is $\mathfrak{S}_{\omega,\Lambda}$ reflexive, or hyper-reflexive?

It would also be interesting to find a description of the C^* -algebras generated by these shifts, $C^*(T_1, \dots, T_n)$. In particular, work on the unweighted case $C^*(S_1, \dots, S_n) \simeq \mathcal{O}_n$ [36] suggests the following problem.

Problem 4.7. Can the C^* -algebras $C^*(T_1, \dots, T_n)$ be described using groupoid techniques?

From a more operator theoretic point of view, we ask if there are extensions of other results on bilateral weighted shift operators to this setting that uncover new phenomena. We are also curious about the

relationship between the notions of periodicity discovered here, and in [25] for versions of unilateral weighted shifts in the noncommutative case.

Problem 4.8. Is there a unifying framework for the noncommutative multivariable notions of periodicity discovered here and in [25]?

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